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Infinitely many solutions for a class of Dirichlet quasilinear elliptic systems

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ABSTRACT

In this paper, we prove the existence of infinitely many classical solutions for a class of Dirichlet quasilinear elliptic systems. The approach is based on variational methods.

1. Introduction

The aim of this paper is to investigate the existence of infinitely many classical solutions for the following Dirichlet quasilinear elliptic system

$$\begin{cases} -(p_i-1)|u_i'(x)|^{p_i-2}u_i''(x) = \lambda F_{u_i}(x,u_1,\ldots,u_n)h_i(x,u_i'), & x\in(a,b),\\ u_i(a)=u_i(b)=0, \quad \text{for } 1\le i\le n, \end{cases} \tag{1.1}$$

where $p_i > 1$ for $1 \le i \le n$, λ is a positive parameter, $a, b \in \mathbb{R}$ with $a < b$, $h_i : [a,b]\times\mathbb{R} \to [0,+\infty)$ is a bounded and continuous function with $m_i := \inf_{(x,t)\in[a,b]\times\mathbb{R}} h_i(x,t) > 0$ for $1 \le i \le n$, $F : [a,b]\times\mathbb{R}^n \to \mathbb{R}$ is a function such that the mapping $(t_1, t_2, \ldots, t_n) \to F(x, t_1, t_2, \ldots, t_n)$ is in C^1 in $\mathbb{R}^n$ for all $x \in [a,b]$, F_{u_i} is continuous in $[a,b]\times\mathbb{R}^n$ for $1 \le i \le n$, where F_{u_i} denotes the partial derivative of F with respect to u_i, and

$$\sup_{|(t_1,\ldots,t_n)|\le M} |F_{u_i}(x,t_1,\ldots,t_n)| \in L^1([a,b])$$

for all $M > 0$ and all $1 \le i \le n$.

Here and in the following, we let X be the Cartesian product of n Sobolev spaces $W_0^{1,p_i}([a,b])$ for $1 \le i \le n$, i.e., $X = \prod_{i=1}^n W_0^{1,p_i}([a,b])$, equipped with the norm

$$\|u\| := \sum_{i=1}^n \|u_i'\|_{p_i}, \quad u=(u_1,u_2,\ldots,u_n),$$

where for $1 \le i \le n$,

$$\|u_i'\|_{p_i} := \left[\int_a^b |u_i'(x)|^{p_i}dx\right]^{\frac{1}{p_i}}.$$

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Then, X is a reflexive real Banach space. Since $p_i > 1$ for $1 \le i \le n$, X is compactly embedded in $C^0([a,b]) \times \cdots \times C^0([a,b])$.

By a classical solution of system (1.1), we mean a function $u = (u_1, \ldots, u_n) \in X$ such that for $1 \le i \le n$, $u_i \in C^1([a,b])$, $u_i' \in AC([a,b])$, and $u_i(t)$ satisfies (1.1) a.e. on $[a,b]$. We say that a function $u = (u_1, \ldots, u_n) \in X$ is a weak solution of system (1.1) if

$$\int_a^b \sum_{i=1}^n \left(\int_0^{u_i'(x)} \frac{(p_i-1)|\tau|^{p_i-2}}{h_i(x,\tau)} d\tau \right) v_i'(x)dx - \lambda \int_a^b \sum_{i=1}^n F_{u_i}(x, u_1(x), \ldots, u_n(x)) v_i(x)dx = 0$$

for all $v = (v_1, \ldots, v_n) \in X$.

The goal of this work is to establish some new criteria for system (1.1) to have infinitely many classical solutions in X. Our analysis is mainly based on a recent critical point theorem of Bonanno and Molica Bisci [1] and is contained in Lemma 2.1 below. This lemma and its variations have been used in several works in order to obtain existence results for different kinds of problems (see, for instance, [2–11] and references therein).

We are motivated by the very recent paper of Graef et al. [12] in which the existence of at least three classical solutions for system (1.1) was established.

Here, as an example, we state a special case of our results.

Theorem 1.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function and $h : \mathbb{R} \to [0, +\infty)$ a bounded and continuous function with $m := \inf_{t\in\mathbb{R}} h(t) > 0$ and $M := \sup_{t\in\mathbb{R}} h(t)$. Put $G(\xi) = \int_0^\xi g(t)dt$ for all $\xi \in \mathbb{R}$ and assume*

$$\liminf_{\xi\to+\infty} \frac{G(\xi)}{\xi^2} < \frac{m}{4M} \limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^2}.$$

Then, for each $\lambda \in \left] \frac{8}{m \limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^2}}, \frac{2}{M \liminf_{\xi\to+\infty} \frac{G(\xi)}{\xi^2}} \right[$, the problem

$$\begin{cases} -u''(x) = \lambda g(u)h(u'), & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.2}$$

has a sequence of classical solutions which is unbounded in $W_0^{1,2}([0,1])$.

For other basic notations and definitions, we refer the reader to [13–16].

2. Main results

In the present section we shall prove our results applying the following smooth version of Theorem 2.1 of [1], which is a more precise version of Ricceri's Variational Principle [17, Theorem 2.5].

Lemma 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\left(\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)},$$
$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \text{and} \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then:

(a) *For every $r > \inf_X \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional*
$$I_\lambda := \Phi - \lambda\Psi$$
to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) *If $\gamma < +\infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either*
- (b_1) *I_λ possesses a global minimum, or*
- (b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*
$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(c) *If $\delta < +\infty$, then for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either*
- (c_1) *there is a global minimum of Φ which is a local minimum of I_λ, or*
- (c_2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ that converges weakly to a global minimum of Φ.*

The following lemma is taken from [12, Lemma 2.2].

Lemma 2.2. *A weak solution to (1.1) in X coincides with a classical solution to (1.1).*

We assume throughout, and without further mention, that the following condition holds:

(H) Either $\underline{p} \geq 2$ or $\overline{p} < 2$, where $\underline{p} := \min\{p_1, \dots, p_n\}$ and $\overline{p} := \max\{p_1, \dots, p_n\}$.

In the following, for $1 \leq i \leq n$, let

$$M_i := \sup_{(x,t)\in[a,b]\times\mathbb{R}} h_i(x,t),$$

$$\overline{M} := \max\{M_i : 1 \leq i \leq n\}, \qquad \underline{m} := \min\{m_i : 1 \leq i \leq n\},$$

and

$$H_i(x,t) := \int_0^t \left(\int_0^\tau \frac{(p_i - 1)|\delta|^{p_i-2}}{h_i(x,\delta)}\, d\delta \right) d\tau$$

for all $(x,t) \in [a,b] \times \mathbb{R}$. Then $\overline{M} \geq M_i \geq m_i \geq \underline{m} > 0$ for each $1 \leq i \leq n$.

Let

$$p^* := \begin{cases} \overline{p}, & \text{if } b - a \geq 1, \\ \underline{p}, & \text{if } 0 < b - a < 1. \end{cases}$$

For all $\gamma > 0$, we set

$$Q(\gamma) := \left\{ (t_1, \dots, t_n) \in \mathbb{R}^n : \sum_{i=1}^n |t_i|^{p_i} \leq \gamma \right\}.$$

Sets of this type will be needed in some of our hypotheses with appropriate choices of γ.

We formulate our main result as follows.

Theorem 2.3. *Assume that there exist two positive constants α and β with $\alpha + \beta < b - a$ such that*

(A_1) *$F(x, t_1, \dots, t_n) \geq 0$ for each $x \in [a, a+\alpha] \cup [b-\beta, b]$ and $(t_1, \dots, t_n) \in \mathbb{R}^n$;*

(A_2) *there exists $1 \leq l \leq n$ such that*

$$0 \leq \liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{(t_1,\dots,t_n)\in Q(\xi^{p_l})} F(x, t_1, \dots, t_n)dx}{\xi^{p_l}} < \left(\frac{2^{\underline{p}}\, \underline{m}}{(b-a)^{p^*-1}\overline{p}\,\overline{M}\, D_l} \right) \limsup_{\xi\to+\infty} \frac{\int_{a+\alpha}^{b-\beta} F(x, 0, \dots, \xi, \dots, 0)dx}{\xi^{p_l}},$$

where

$$D_l := \frac{(p_l - 1)^{p_l-2}}{p_l} (\alpha^{-p_l+1} + \beta^{-p_l+1}), \tag{2.1}$$

and in $F(x, 0, \dots, \xi, \dots, 0)$, ξ is the $(l+1)$-th argument.

Then, for each $\lambda \in \Lambda$, system (1.1) has an unbounded sequence of classical solutions, where

$$\Lambda := \left] \frac{D_l}{\underline{m} \limsup_{\xi\to+\infty} \frac{\int_{a+\alpha}^{b-\beta} F(x,0,\dots,\xi,\dots,0)dx}{\xi^{p_l}}}, \frac{2^{\underline{p}}}{(b-a)^{p^*-1}\overline{p}\,\overline{M} \liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{(t_1,\dots,t_n)\in Q(\xi^{p_l})} F(x,t_1,\dots,t_n)dx}{\xi^{p_l}}} \right[. \tag{2.2}$$

Proof. Our aim is to apply Lemma 2.1(b) to our problem. To this end, for each $u = (u_1, \dots, u_n) \in X$, we let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) := \sum_{i=1}^n \int_a^b H_i(x, u_i'(x))dx \tag{2.3}$$

and

$$\Psi(u) := \int_a^b F(x, u_1(x), \dots, u_n(x))dx. \tag{2.4}$$

It is well known that Φ and Ψ are well defined and continuously differentiable functionals whose derivatives at the point $u = (u_1, \dots, u_n) \in X$ are the functionals $\Phi'(u), \Psi'(u) \in X^*$ given by

$$\Phi'(u)(v) = \int_a^b \sum_{i=1}^n \left(\int_0^{u_i'(x)} \frac{(p_i-1)|\tau|^{p_i-2}}{h_i(x,\tau)} d\tau \right) v_i'(x)dx$$

and

$$\Psi'(u)(v) = \int_a^b \sum_{i=1}^n F_{u_i}(x, u_1(x), \dots, u_n(x)) v_i(x)dx$$

for every $v = (v_1, \dots, v_n) \in X$. Since Φ' is monotone (see the proof of [12, Lemma 2.1]), by applying Proposition 25.20 of [16], Φ is sequentially weakly lower semicontinuous. Moreover, Φ is strongly continuous and Ψ is weakly upper semicontinuous.

Also, since $0 < h_i(x,t) \le \overline{M}$ for $1 \le i \le n$ and $(x,t) \in [a,b] \times \mathbb{R}$, from (2.3), we see that for all $u = (u_1, \dots, u_n) \in X$,

$$\Phi(u) \ge \frac{1}{\overline{M}} \sum_{i=1}^n \frac{\|u_i'\|_{p_i}^{p_i}}{p_i} \ge \frac{1}{\overline{p}\overline{M}} \sum_{i=1}^n \|u_i'\|_{p_i}^{p_i}, \tag{2.5}$$

and so Φ is coercive.

Let $\{\xi_k\}$ be a sequence of positive numbers such that $\xi_k \to +\infty$ as $k \to +\infty$ and

$$\lim_{k\to+\infty} \frac{\int_a^b \sup_{(t_1,\dots,t_n)\in Q(\xi_k^{p_l})} F(x,t_1,\dots,t_n)dx}{\xi_k^{p_l}} = \liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{(t_1,\dots,t_n)\in Q(\xi^{p_l})} F(x,t_1,\dots,t_n)dx}{\xi^{p_l}}. \tag{2.6}$$

At this point, we have (see [18])

$$\max_{x\in[a,b]} |u_i(x)| \le \frac{(b-a)^{\frac{p_i-1}{p_i}}}{2} \|u_i'\|_{p_i}$$

for $1 \le i \le n$. Then,

$$\max_{x\in[a,b]} \sum_{i=1}^n |u_i(x)|^{p_i} \le \frac{(b-a)^{p^*-1}}{2^{\underline{p}}} \sum_{i=1}^n \|u_i'\|_{p_i}^{p_i}$$

for $1 \le i \le n$.

Let

$$r_k := \frac{2^{\underline{p}}\, \xi_k^{p_l}}{(b-a)^{p^*-1}\overline{p}\,\overline{M}}$$

for $k \in \mathbb{N}$. Then, for $v = (v_1, \dots, v_n) \in X$ with $\Phi(v) < r_k$, together with (2.5), we have

$$\max_{x\in[a,b]} \sum_{i=1}^n |v_i(x)|^{p_i} \le \xi_k^{p_l}.$$

Note that $0 \in \Phi^{-1}(-\infty, r_k)$ and $\Psi(0) \ge 0$ by (A_1). Then,

$$\begin{aligned}
\varphi(r_k) &= \inf_{u\in\Phi^{-1}(-\infty,r_k)} \frac{\left(\sup_{v\in\Phi^{-1}(-\infty,r_k)} \Psi(v)\right) - \Psi(u)}{r_k - \Phi(u)} \\
&\le \frac{\sup_{v\in\Phi^{-1}(-\infty,r_k)} \Psi(v)}{r_k} \\
&\le \frac{(b-a)^{p^*-1}\overline{p}\,\overline{M} \int_a^b \sup_{(t_1,\dots,t_n)\in Q(\xi_k^{p_l})} F(x,t_1,\dots,t_n)dx}{2^{\underline{p}}\, \xi_k^{p_l}}.
\end{aligned}$$

Therefore, from (2.6) and (A_2) we have

$$\begin{aligned}
\gamma &= \liminf_{k\to+\infty} \varphi(r_k) \\
&\le \frac{(b-a)^{p^*-1}\overline{p}\,\overline{M}}{2^{\underline{p}}} \liminf_{\xi\to+\infty} \frac{\int_a^b \sup_{(t_1,\dots,t_n)\in Q(\xi^{p_l})} F(x,t_1,\dots,t_n)dx}{\xi^{p_l}} < +\infty.
\end{aligned} \tag{2.7}$$

Due to (A_2), (2.2) and (2.7), we see that $\Lambda \subseteq (0, 1/\gamma)$. Let $\lambda \in \Lambda$ be fixed. We claim that the functional $I_\lambda = \Phi - \lambda\Psi$ is unbounded from below. Since $\lambda \in \Lambda$, by (2.2), we have

$$\frac{1}{\lambda} < \frac{\underline{m}}{D_l} \limsup_{\xi \to +\infty} \frac{\int_{a+\alpha}^{b-\beta} F(x, 0, \ldots, \xi, \ldots, 0)dx}{\xi^{p_l}}.$$

Then, there exists a sequence $\{d_k\}$ of positive numbers and a constant τ such that $d_k \to +\infty$ as $k \to +\infty$ and

$$\frac{1}{\lambda} < \tau < \frac{\underline{m}}{D_l} \frac{\int_{a+\alpha}^{b-\beta} F(x, 0, \ldots, d_k, \ldots, 0)dx}{d_k^{p_l}} \tag{2.8}$$

for $k \in \mathbb{N}$ large enough. Let $\{w_k\}$ be a sequence in X defined by $w_k(x) = (0, \ldots, w_{lk}(x), \ldots, 0)$, where w_{lk} is the l-th argument of w_k and is defined by

$$w_{lk}(x) := \begin{cases} \dfrac{1}{\alpha^{p_l-1}} d_k(x-a)^{p_l-1}, & \text{if } a \le x < a+\alpha, \\ d_k, & \text{if } a+\alpha \le x \le b-\beta, \\ \dfrac{1}{\beta^{p_l-1}} d_k(b-x)^{p_l-1}, & \text{if } b-\beta < x \le b. \end{cases}$$

For any fixed $k \in \mathbb{N}$, clearly $w_k = (0, \ldots, w_{lk}, \ldots, 0) \in X$ and

$$\Phi(w_k) \le \frac{d_k^{p_l} D_l}{\underline{m}}. \tag{2.9}$$

From (A_1) and (2.4), we have

$$\Psi(w_k) \ge \int_{a+\alpha}^{b-\beta} F(x, 0, \ldots, d_k, \ldots, 0)dx. \tag{2.10}$$

By (2.8)–(2.10), we see that

$$\begin{aligned} \Phi(w_k) - \lambda\Psi(w_k) &\le \frac{d_k^{p_l} D_l}{\underline{m}} - \lambda \int_{a+\alpha}^{b-\beta} F(x, 0, \ldots, d_k, \ldots, 0)dx \\ &< \frac{d_k^{p_l} D_l}{\underline{m}}(1 - \lambda\tau) \end{aligned}$$

for every $k \in \mathbb{N}$ large enough. Since $\lambda\tau > 1$ and $d_k \to +\infty$ as $k \to +\infty$, we have

$$\lim_{k \to +\infty} \big(\Phi(w_k) - \lambda\Psi(w_k)\big) = -\infty.$$

Then, the functional $I_\lambda = \Phi - \lambda\Psi$ is unbounded from below. Therefore, by Lemma 2.1(b), there exists a sequence $\{(u_{1k}, \ldots, u_{nk})\}$ of critical points of I_λ such that

$$\lim_{k \to +\infty} \|(u_{1k}, \ldots, u_{nk})\| = +\infty.$$

Also, note that the weak solutions of (1.1) are exactly critical points of I_λ. So, applying Lemma 2.2, the conclusion follows from Lemma 2.1. □

We now point out the following special case of Theorem 2.3 when F does not depend on $x \in [a, b]$.

Corollary 2.4. *Assume that there exist two positive constants α and β with $\alpha + \beta < b - a$ such that*

(B_1) *$F(t_1, \ldots, t_n) \ge 0$ for each $(t_1, \ldots, t_n) \in \mathbb{R}^n$;*
(B_2) *there exists $1 \le l \le n$ such that*

$$\liminf_{\xi \to +\infty} \frac{\sup_{(t_1,\ldots,t_n) \in Q(\xi^{p_l})} F(t_1, \ldots, t_n)}{\xi^{p_l}} < \left(\frac{(b-a) - (\alpha+\beta)}{(b-a)^{p^*}} \frac{2^{\underline{p}}\, \underline{m}}{\underline{p}\, \overline{M}\, D_l} \right) \limsup_{\xi \to +\infty} \frac{F(0, \ldots, \xi, \ldots, 0)}{\xi^{p_l}},$$

where D_l is given by (2.1) and in $F(0, \ldots, \xi, \ldots, 0)$, ξ is the l-th argument.

Then, for each

$$\lambda \in \left]\frac{D_l}{\underline{m}\Big((b-a)-(\alpha+\beta)\Big)\limsup\limits_{\xi\to+\infty}\frac{F(0,\ldots,\xi,\ldots,0)}{\xi^{p_l}}}, \frac{2^{\underline{p}}}{(b-a)^{p^*}\,\overline{p}\,\overline{M}\,\liminf\limits_{\xi\to+\infty}\frac{\sup_{(t_1,\ldots,t_n)\in Q(\xi^{p_l})}F(t_1,\ldots,t_n)}{\xi^{p_l}}}\right[,$$

the system

$$\begin{cases}-(p_i-1)|u_i'(x)|^{p_i-2}u_i''(x)=\lambda F_{u_i}(u_1,\ldots,u_n)h_i(x,u_i'), & x\in(a,b),\\ u_i(a)=u_i(b)=0, \quad \text{for } 1\le i\le n,\end{cases}$$

has an unbounded sequence of classical solutions.

Remark 2.5. Theorem 1.1 in the Introduction immediately follows from Corollary 2.4.

Now, we present an example of the application of Theorem 1.1.

Example 2.6. Put

$$a_n := \frac{2n!(n+2)!-1}{4(n+1)!}, \qquad b_n := \frac{2n!(n+2)!+1}{4(n+1)!}$$

for every $n\in\mathbb{N}$, and define the nonnegative continuous function $g:\mathbb{R}\to\mathbb{R}$ by

$$g(\xi)=\begin{cases}\dfrac{32(n+1)!^2[(n+1)!^2-n!^2]}{\pi}\sqrt{\dfrac{1}{16(n+1)!^2}-\left(\xi-\dfrac{n!(n+2)}{2}\right)^2}, & \text{if } \xi\in\bigcup\limits_{n\in\mathbb{N}}[a_n,b_n],\\ 0, & \text{otherwise.}\end{cases}$$

One has

$$\int_{n!}^{(n+1)!} g(t)dt=\int_{a_n}^{b_n} g(t)dt=(n+1)!^2-n!^2$$

for every $n\in\mathbb{N}$. Then, one has

$$\lim_{n\to+\infty}\frac{G(a_n)}{a_n^2}=0 \quad\text{and}\quad \lim_{n\to+\infty}\frac{G(b_n)}{b_n^2}=4.$$

Therefore, by a simple computation, we obtain

$$\liminf_{\xi\to+\infty}\frac{G(\xi)}{\xi^2}=0 \quad\text{and}\quad \limsup_{\xi\to+\infty}\frac{G(\xi)}{\xi^2}=4.$$

Also, let

$$h(t)=\begin{cases}1, & \text{if } t<0,\\ \dfrac{1}{t+1}, & \text{if } 0\le t\le 1,\\ \dfrac{1}{2}, & \text{if } t>1.\end{cases}$$

Then, $h:\mathbb{R}\to[0,+\infty)$ is a bounded and continuous function with

$$m=\inf_{t\in\mathbb{R}}h(t)=\frac{1}{2}>0 \quad\text{and}\quad M=\sup_{t\in\mathbb{R}}h(t)=1.$$

We have

$$0=\liminf_{\xi\to+\infty}\frac{G(\xi)}{\xi^2}<\frac{m}{4M}\limsup_{\xi\to+\infty}\frac{G(\xi)}{\xi^2}=\frac{1}{2}.$$

So, from Theorem 1.1, for each $\lambda>4$, problem (1.2) has a sequence of classical solutions which is unbounded in $W_0^{1,2}([0,1])$.

Corollary 2.7. *Assume that for $1\le i\le n$, $g_i:\mathbb{R}\to\mathbb{R}$ are continuously differentiable functions and there exist two positive constants α and β with $\alpha+\beta<b-a$ such that*

(C_1) *$g_i(t) \geq 0$ for each $1 \leq i \leq n$ and $t \in \mathbb{R}$;*
(C_2)

$$\liminf_{\xi\to+\infty} \frac{\sup_{(t_1,\ldots,t_n)\in Q(\xi^{p_n})} \prod_{i=1}^{n} g_i(t_i)}{\xi^{p_n}} < \left(\frac{(b-a)-(\alpha+\beta)}{(b-a)^{p^*}} \frac{2^{\underline{p}}\,\underline{m}}{\overline{p}\,\overline{M}\,D_n}\right) \prod_{i=1}^{n-1} g_i(0) \limsup_{\xi\to+\infty} \frac{g_n(\xi)}{\xi^{p_n}},$$

where

$$D_n := \frac{(p_n-1)^{p_n-2}}{p_n}\left(\alpha^{-p_n+1} + \beta^{-p_n+1}\right).$$

Then, for each

$$\lambda \in \left] \frac{D_n}{\underline{m}\Big((b-a)-(\alpha+\beta)\Big) \prod_{i=1}^{n-1} g_i(0) \limsup_{\xi\to+\infty} \frac{g_n(\xi)}{\xi^{p_n}}}, \frac{2^{\underline{p}}}{(b-a)^{p^*}\,\overline{p}\,\overline{M} \liminf_{\xi\to+\infty} \frac{\sup_{(t_1,\ldots,t_n)\in Q(\xi^{p_n})} \prod_{i=1}^{n} g_i(t_i)}{\xi^{p_n}}} \right[,$$

the system

$$\begin{cases} -(p_i-1)|u_i'(x)|^{p_i-2}u_i''(x) = \lambda\, g_i'(u_i)\Big(\prod_{j=1_{j\neq i}}^{n} g_j(u_j)\Big) h_i(x, u_i'), & x \in (a,b), \\ u_i(a) = u_i(b) = 0, \quad \text{for } 1 \leq i \leq n, \end{cases}$$

has an unbounded sequence of classical solutions.

Finally, we want to point out a simple consequence of Corollary 2.4.

Corollary 2.8. *Assume that Assumption* (B_1) *in Corollary 2.4 holds, and there exists $1 \leq l \leq n$ such that*

$$\liminf_{\xi\to+\infty} \frac{\sup_{(t_1,\ldots,t_n)\in Q(\xi^{p_l})} F(t_1,\ldots,t_n)}{\xi^{p_l}} = 0$$

and

$$\limsup_{\xi\to+\infty} \frac{F(0,\ldots,\xi,\ldots,0)}{\xi^{p_l}} = +\infty,$$

where, in $F(0,\ldots,\xi,\ldots,0)$, ξ is the l-th argument. Then, the system

$$\begin{cases} -(p_i-1)|u_i'(x)|^{p_i-2}u_i''(x) = F_{u_i}(u_1,\ldots,u_n) h_i(x, u_i'), & x \in (a,b), \\ u_i(a) = u_i(b) = 0, \quad \text{for } 1 \leq i \leq n, \end{cases}$$

has an unbounded sequence of classical solutions.

Remark 2.9. In Theorem 2.3 we can replace $\xi \to +\infty$ by $\xi \to 0^+$, applying in the proof part (c) of Lemma 2.1 instead of (b). In this case we obtain a sequence of pairwise distinct classical solutions to the system (1.1) which converges uniformly to zero. Also, results similar to Corollaries 2.4, 2.7 and 2.8 and Theorem 1.1 can be obtained.

References

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